Thermal Instability of a Compressible Hall Plasma

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The thermal instability of a Hall plasma taking compressibility into account is studied. The effect of compressibility is found to be stabilizing while the Hall currents have a destabilizing effect. The system is stable for $(c_p/g)\beta<1$. Oscillatory modes are introduced due to the presence of a magnetic field which are missing in the absence of a magnetic field (and hence in the absence of Hall currents) for $(c_p/g)\beta>1$. The case of overstability is also discussed.

1. Introduction

The problem of thermal instability of fluids under various assumptions of hydrodynamics and hydromagnetics has been summarised by Chandrasekhar [1]. Gupta [2] studied the effect of Hall currents on the thermal instability of a horizontal layer of a conducting fluid. The Boussinesq approximation has been used in the above studies.

Spiegel and Veronis [3] have simplified the set of equations governing the flow of compressible fluids under the following assumptions:

(i) the depth of fluid layer is much smaller than the scale height as defined by them and

(ii) the fluctuations in temperature, pressure and density, introduced due to motion, do not exceed their static variations.

Under the above assumptions, Spiegel and Veronis [3] have found the flow equations to be the same as for incompressible fluids except that the static temperature gradient is replaced by its excess over the adiabatic one.

A reconsideration of the thermal instability problem in the presence of compressibility and Hall currents is certainly called for, as these effects are likely to be important in the ionosphere and outer layers of the sun's atmosphere. This forms the subject matter of the present paper.

2. Formulation of the Problem and Dispersion Relation

The problem and the configuration is the same as that studied by Gupta [2] except that the fluid is compressible here.

Spiegel and Veronis [3] expressed any space variable, say X, in the form

$$X = X_{\mathrm{m}} + X_0(z) + X'(x, y, z, t), \tag{1}$$

where X_{m} stands for the constant space distribution of X, X_0 is the variation in X in the absence of motion and $X'(x, y, z, t)$ stands for the fluctuations in X due to the motion of the fluid.

The initial state is one in which the velocity, temperature, pressure and density at any point in the fluid are respectively given by

$$\boldsymbol{q} = 0, \quad T = T(z), \quad p = p(z), \quad \varrho = \varrho(z), \tag{2}$$

where according to Spiegel and Veronis [3]

$$\begin{aligned} T(z) &= -\beta z + T_0, \\ p(z) &= p_{\mathrm{m}} - g \int_0^z (\varrho_{\mathrm{m}} + \varrho_0)\,\mathrm{d}z, \\ \varrho(z) &= \varrho_{\mathrm{m}}[1 - \alpha_{\mathrm{m}}(T - T_{\mathrm{m}}) + K_{\mathrm{m}}(p - p_{\mathrm{m}})], \\ \alpha_{\mathrm{m}} &= -\left(\frac{1}{\varrho}\frac{\partial \varrho}{\partial T}\right)_m, \quad K_{\mathrm{m}} = \left(\frac{1}{\varrho}\frac{\partial \varrho}{\partial p}\right)_m. \end{aligned} \tag{3}$$

Let μ, $\nu\,(=\mu/\varrho_{\mathrm{m}})$, $\varkappa'$, $\varkappa\,(=\varkappa'/\varrho_{\mathrm{m}} c_p)$ and μ_{e} denote respectively the viscosity, the kinematic viscosity, the thermal conductivity, the thermal diffusivity and the magnetic permeability. $\alpha_{\mathrm{m}}\,(=\alpha$, say) is the coefficient of thermal expansion.

The relevant equations of the problem under consideration, in nondimensional form, are

$$(D^2 - a^2)(D^2 - a^2 - \sigma) W - \left(\frac{g\,\alpha\,d^2}{\nu}\right) a^2 \Theta + \frac{\mu_{\mathrm{e}} H d}{4\pi\varrho_{\mathrm{m}}\nu}(D^2 - a^2) D K = 0, \tag{4}$$

$$(D^2 - a^2 - \sigma) Z = -\left(\frac{\mu_{\mathrm{e}} H d}{4\pi\varrho_{\mathrm{m}}\nu}\right) D X, \tag{5}$$

$$(D^2 - a^2 - p_2\sigma) X = -\left(\frac{H d}{\eta}\right) D Z - \left(\frac{H}{4\pi N e \eta d}\right)(D^2 - a^2) D K, \tag{6}$$

$$(D^2 - a^2 - p_2\sigma) K = -\left(\frac{H d}{\eta}\right) D W + \left(\frac{H d}{4\pi N e \eta}\right) D X, \tag{7}$$

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$$(D^2 - a^2 - p_1 \sigma)\,\Theta = -\frac{d^2}{\varkappa}\left(\beta - \frac{g}{c_p}\right) W\,, \tag{8}$$

where g/c_p and $\varkappa$ stand for the adiabatic gradient and the thermal diffusivity respectively. c_p is the specific heat at constant pressure. N and e denote the electron number density and electron charge respectively. Other symbols have their usual meanings.

We consider the case in which the boundaries are free and the medium adjoining the fluid is nonconducting. The boundary conditions appropriate for the problem are (Chandrasekhar [1]):

$$\left.\begin{array}{l} W = D^2 W = 0\,, \quad \Theta = 0\,, \quad D Z = 0 \\ X = 0 \text{ and } \mathbf{K} \text{ are continuous} \end{array}\right\} \text{ at } z = 0 \text{ and } 1. \tag{9}$$

In the absence of any surface current, the tangential components of the magnetic field are continuous. Hence the boundary conditions in addition to (9) are

$$D K = 0\,, \tag{10}$$

on the boundaries. Eliminating Z, K, X and Θ between Eqs. (4) – (8) and substituting the proper solution $W = W_0 \sin \pi z$, W_0 being a constant, in the resultant equation, we obtain the dispersion relation

$$R_1 x = \left(\frac{G}{G-1}\right)\Bigg[(1+x)\left(1+x+\frac{\sigma}{\pi^2}\right)\left(1+x+p_1\frac{\sigma}{\pi^2}\right) + \frac{Q_1(1+x)\left(1+x+p_1\frac{\sigma}{\pi^2}\right)\left\{\left(1+x+\frac{\sigma}{\pi^2}\right)\left(1+x+p_2\frac{\sigma}{\pi^2}\right)+Q_1\right\}}{\left(1+x+p_2\frac{\sigma}{\pi^2}\right)\left\{\left(1+x+\frac{\sigma}{\pi^2}\right)\left(1+x+p_2\frac{\sigma}{\pi^2}\right)+Q_1\right\}+M(1+x)\left(1+x+\frac{\sigma}{\pi^2}\right)}\Bigg], \tag{11}$$

where

$$Q_1 = \frac{\mu_e H^2 d^2}{4\pi \varrho_m \nu \eta \pi^2}\,, \quad R_1 = \frac{g \alpha \beta d^4}{\nu \varkappa \pi^4}\,, \quad M = \left(\frac{H}{4\pi N e \eta}\right)^2 \quad \text{and} \quad G = \frac{c_p}{g}\beta\,.$$

3. Stability of the System and Oscillatory Modes

Multiplying Eq. (4) by W^*, the complex conjugate of W, and using Eqs. (5) – (8) together with boundary conditions (9) and (10), we obtain

$$I_1 + \sigma I_2 + \frac{\mu_e \eta}{4\pi \varrho_m \nu}(I_5 + p_2 \sigma^* I_6) + \frac{\mu_e \eta}{4\pi \varrho_m \nu} d^2 (I_7 + p_2 \sigma I_8) + d^2 (I_9 + \sigma^* I_{10}) = \frac{c_p \alpha \varkappa a^2}{\nu (G-1)}(I_3 + p_1 \sigma^* I_4)\,, \tag{12}$$

where

$$\begin{aligned}
I_1 &= \int_0^1 (|D^2 W|^2 + 2a^2 |D W|^2 + a^4 |W|^2)\,dz\,, \quad I_2 = \int_0^1 (|D W|^2 + a^2 |W|^2)\,dz\,,\\
I_3 &= \int_0^1 (|D \Theta|^2 + a^2 |\Theta|^2)\,dz\,, \quad I_4 = \int_0^1 |\Theta|^2\,dz\,,\\
I_5 &= \int_0^1 (|D^2 K|^2 + 2a^2 |D K|^2 + a^4 |K|^2)\,dz\,, \quad I_6 = \int_0^1 (|D K|^2 + a^2 |K|^2)\,dz\,,\\
I_7 &= \int_0^1 (|D X|^2 + a^2 |X|^2)\,dz\,, \quad I_8 = \int_0^1 |X|^2\,dz\,,\\
I_9 &= \int_0^1 (|D Z|^2 + a^2 |Z|^2)\,dz\,, \quad I_{10} = \int_0^1 |Z|^2\,dz\,,
\end{aligned} \tag{13}$$

which are all positive definite.

Putting $\sigma = \sigma + i\sigma_i$ and then equating real and imaginary parts of Eq. (12), we arrive at the following conclusions:

Theorem 1. If $G < 1$, the system is stable.

Theorem 2. In contrast to the nonoscillatory modes for $G > 1$ in the absence of a magnetic field, the presence of a magnetic field (and hence the presence of Hall effect) introduces oscillatory modes in the system.

4. The Stationary Convection

For stationary convection, putting $\sigma = 0$ in Eq. (11) reduces it to

$$R_1 = \left(\frac{1+x}{x}\right)\left(\frac{G}{G-1}\right)\left[\frac{\{(1+x)^2+Q_1\}^2+M(1+x)^3}{(1+x)^2+Q_1+M(1+x)}\right]. \tag{14}$$

For fixed values of Q_1 and M, let the nondimensional number G accounting for the compressibility effects be also kept as fixed, then we find that

$$\overline{R}_c = \left(\frac{G}{G-1}\right) R_c\,, \tag{15}$$

where $\overline{R}_c$ and R_c denote respectively the critical Rayleigh numbers in the presence and absence of compressibility. The effect of compressibility is, thus, to postpone the onset of thermal instability. Hence we obtain a stabilizing effect of compressibility.

From Eq. (14), it follows that

$$\frac{dR_1}{dM} = -\frac{\left(\frac{1+x}{x}\right)\left(\frac{G}{G-1}\right) Q_1(1+x)\,[(1+x)^2+Q_1]}{[(1+x)^2+Q_1+M(1+x)]^2}\,,$$

which is always negative. This shows that Hall currents have a destabilizing effect on the system.

5. The Overstable Case

Put $\sigma/\pi^2 = i\,\sigma_1$, where σ_1 is real, in Equation (11). Equating real and imaginary parts of the resultant equation and eliminating R_1 between them, one obtains

$$A_1\beta'^3 + B_1\beta'^2 + C_1\beta' + D_1 = 0\,, \quad \text{where} \quad A_1 = p_2^4(1+p_1)\alpha'\,, \quad \beta' = \sigma_1^2, \quad 1+x = \alpha', \tag{16}$$

and $D_1 = (1+p_1)\alpha'^7 + 2M(1+p_1)\alpha'^6 + [(3p_1 - p_2 + 2)Q_1 + M^2(1+p_1)]\alpha'^5 + M Q_1(3p_1 + p_2 + 2)\alpha'^4$
$+ (3p_1 - 2p_2 + 1)Q_1^2\alpha'^3 + M Q_1^2(p_1 - 1)\alpha'^2 + Q_1^3(p_1 - p_2)\alpha'\,.$

Following the arguments as in Gupta [2], we find that for $\varkappa < \nu$ and $\varkappa < \eta$, overstability is not possible and the principle of exchange of stabilities is valid. $\varkappa < \nu$ and $\varkappa < \eta$ are therefore sufficient conditions for nonexistence of overstability.

[1] S. Chandrasekhar, Hydrodynamic and Hydromagnetic Stability, Chap. IV, Clarendon Press, Oxford 1961.

[2] A. S. Gupta, Rev. Roumaine Math. Pures Appl. **12**, 665 [1967].

[3] E. A. Spiegel and G. Veronis, Astrophys. J. **131**, 442 [1960].